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Letter Section

On the existence of conjugate points for a constrained minimization problem

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Abstract: An isoparametric problem related to a direct current motor is studied. The solution candidate is tested for possible conjugate points by means of the theory of Weierstrass. Contrary to some previous conclusions, no conjugate points seem to exist.

We consider a direct current electric motor obeying the equation

$$i = \frac{d\nu}{d\tau} + \mu = \nu' + \mu, \tag{1}$$

where τ equals time divided by a constant T_m, $i = i(\tau)$ is the current, $\nu = \nu(\tau)$ the angular velocity of the armature, and μ the loading torque [2].

Our problem is to minimize the losses

$$L = \int_0^T i^2 d\tau \tag{2}$$

in the armature, when T and the total angle of rotation

$$\alpha = \int_0^T \nu d\tau \tag{3}$$

are given together with the boundary conditions $\nu(0) = \nu(T) = 0$ for ν.

We recognize the problem as a typical isoperimetric problem in the calculus of variations with $\nu(\tau)$ as the unknown function. The family of extremals is easily found to be

$$\nu(\tau; \lambda, a_1, a_2) = \tfrac{1}{4}\lambda\tau^2 + a_2\tau + a_1, \tag{4}$$

where a_1 and a_2 are constants and λ is the usual Lagrange mutiplier.

Using the boundary values and the condition (3), we obtain

$$\lambda = -24\alpha/T^3, \tag{5}$$

$$a_1 = 0 \quad \text{and} \quad a_2 = 6\alpha/T^2. \tag{6}$$

Our solution candidate is thus

$$\nu(\tau) = (6\alpha/T^2)(\tau - \tau^2/T). \tag{7}$$

In order to investigate (7) for conjugate points we apply the results of Weierstrass as explained in [1]. We accommodate to the treatment in [1] by casting (2) and (3) into parametric form:

$$L = \int_0^t (\nu'/\tau' + \mu)^2 \tau' dt, \tag{2'}$$

$$\alpha = \int_0^t \nu\tau' dt, \tag{3'}$$

where t denotes a general parameter, and compute Bolza's expression U (see [1, pp. 215, 219, 220]):

$$G(\nu, \tau, \nu', \tau') = \nu\tau',$$

$$\begin{aligned} U &= G_{\tau\nu'} - G_{\tau'\nu} - (G_{\tau'\tau'}/\tau'^2)(\tau'\nu'' - \tau''\nu') \\ &= 0 - 1 - O(\tau'v'' - t''v') = -1. \end{aligned}$$

As the functions f and g [1, p. 213] we can take (our ν and τ correspond to the functions y and x of [1])

$$\nu(\tau) = \tfrac{1}{4}\lambda t^2 + a_2 t + a_1 \equiv g(t, a_2, a_1, \lambda), \tag{8}$$

$$\tau(t) = t \equiv f(t, a_2, a_1, \lambda), \tag{9}$$

i.e. we use τ as the parameter t. By means of (8) and (9) we further compute $\theta_1(t) = -t$, $\theta_2(t) = -1$, $\theta_3(t) = -\frac{1}{4}t^2$, and finally

$$D(t, t_0) = \begin{vmatrix} 0 & -1 & 0 \\ -t & -1 & -\frac{1}{4}t^2 \\ \frac{1}{2}t^2 & t & \frac{1}{12}t^3 \end{vmatrix} = \tfrac{1}{24}t^4. \tag{10}$$

Since $D(t, t_0) \neq 0$ for $t \neq 0$ $(= t_0)$, our candidate passes the conjugate point test. Consequently, it might well, contrary to the conclusions of [2], be an acceptable minimizing curve.

References

[1] O. Bolza, *Calculus of Variations* (Chelsea, New York, 1973).
[2] M. Razzaghi, On the Jacobi condition for optimizing an engineering problem, *J. Comput. Appl. Math.* **2** (2) (1976) 77–80.